MA 441, Fall 2011

Assignment 10.

This homework is due *Tuesday* Nov 22.

There are total 47 points in this assignment. 42 points is considered 100%. If you go over 42 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 5.4, 5.6 and 6.1 in Bartle–Sherbert.

(1) [3pt] (Exercise 5.4.2) Show that function $f(x) = 1/x^2$ is uniformly continuous on $A = [1, \infty)$, but that it is not uniformly continuous on $B = (0, \infty)$.

(2) (Exercise 5.4.3) Use the Nonuniform Continuity Criterion to show that the following functions are not uniformly continuous on the given sets.
  (a) [2pt] $f(x) = x^2$, $A = [0, \infty)$.
  (b) [2pt] $g(x) = \sin(1/x)$, $B = (0, \infty)$.

(3) (a) [4pt] (Exercise 5.4.6) Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$, and if they are *both* bounded on A, then their product fg is uniformly continuous on A.
  (b) [3pt] Give an example of the set $A \subseteq \mathbb{R}$ and two functions $f, g : A \to \mathbb{R}$ uniformly continuous on A such that their product fg is not uniformly continuous on A. (*Hint:* $x^2 = x \cdot x$. Work out details.)
  Comment. In fact, only one of functions needs to be unbounded to deliver non-uniform continuity of product. For example, product $x \cdot \sin x$ is not uniformly continuous on $\mathbb{R}$. You are not asked to prove this, though.

(4) (a) [3pt] (5.6.4) If f and g are *positive* increasing functions on an interval $I \subseteq \mathbb{R}$, then their product fg is increasing on I.
  (b) [2pt] (5.6.3) Show that both $f(x) = x$, $g(x) = x - 1$ are strictly increasing on $I = [0, 1]$, but that their product fg is not increasing on I.

(5) (Part of 6.1.1) Use the definition to find derivative of each of the following functions:
  (a) [2pt] $f(x) = 1/x$, $x \in \mathbb{R}$, $x \neq 0$
  (b) [2pt] $f(x) = \sqrt{x}$, $x > 0$.
  (*Hint:* You can use any of three definitions of the derivative that were discussed in class, but the shortest here probably is the limit of ratio definition.)

— *see next page* —

(6) [3pt] (6.1.2) Show that $f(x) = x^{1/3}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

(7) [4pt] (6.1.4) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for x rational, $f(x) = 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$. (Hint: Use the limit definition of derivative.)

(8) (Part of 6.1.8) Determine where each of the following functions from $\mathbb{R}$ to $\mathbb{R}$ is differentiable and find the derivative:
(a) [3pt] $f(x) = x|x|$.
(b) [3pt] $g(x) = 2x + |x|$.

(9) [4pt] (6.1.9) Reminder. Function $f : \mathbb{R} \to \mathbb{R}$ is called
even if $\forall x \in \mathbb{R}, f(-x) = f(x)$;
odd if $\forall x \in \mathbb{R}, f(-x) = -f(x)$.
Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function and has a derivative at every point, then the derivative f' is an odd function. Also prove that if $g : \mathbb{R} \to \mathbb{R}$ is a differentiable odd function, then g' is an even function. (*Hint:* Compute $f'(-x), g'(-x)$.)

(10) [4pt] (6.1.10) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$, and $g(0) = 0$. Show that g is differentiable for all $x \in \mathbb{R}$. Also show that the derivative g' is not bounded on the interval $[-1, 1]$.

(11) [3pt] (6.1.15) Given that the restriction of the cosine function cos to $I = [0, \pi]$ is strictly decreasing and that $\cos 0 = 1$, $\cos \pi = -1$, let $J = [-1, 1]$, and let $\arccos : J \to \mathbb{R}$ be the function inverse to the restriction of cos to I. Show that arccos is differentiable on $(-1, 1)$ and $(\arccos y)' = (-1)/(1 - y^2)^{1/2}$ for $y \in (-1, 1)$. Show that arccos is not differentiable at -1 and 1.